Multiple Choice
Please look over the formula sheet before you start your exam.

1. The function

$$f(x) = \frac{1}{x} - \ln(x) + 3$$

is a one-to-one function (there is no need to check this). What is $(f^{-1})'(4)$?

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{4}{5}$ (d) 2 (e) $-\frac{5}{4}$

First find x such that $f(x) = 4$. Try a few values: 0 not in domain, 1 gives $f(1) = 4$. Hence $(f^{-1})' = \frac{1}{f'(1)}$. Since $f'(x) = -x^{-2} = x^{-1} + 0$, $f'(1) = -1 - 1 = -2$ so $(f^{-1})'(4) = -\frac{1}{2}$.

2. Find $f'(x)$, where

$$f(x) = \frac{(x^3+1)\sqrt{2x+3}}{\cos^2 x}$$

Hint: Logarithmic differentiation may be helpful.

(a) $\dfrac{(x^3+1)\sqrt{2x+3}}{\cos^2 x}\left(\dfrac{3x^2}{x^3+1}+\dfrac{1}{2x+3}+\dfrac{2\sin x}{\cos x}\right)$

(b) $\dfrac{(x^3+1)\sqrt{2x+3}}{\cos^2 x}\left(\dfrac{1}{x^3+1}+\dfrac{1}{2(2x+3)}-\dfrac{2}{\cos x}\right)$

(c) $\dfrac{(x^3+1)\sqrt{2x+3}}{\cos^2 x}\left(\dfrac{3x^2}{x^3+1}+\dfrac{1}{2(2x+3)}-\dfrac{2\sin x}{\cos x}\right)$

(d) $\dfrac{(x^3+1)\sqrt{2x+3}}{\cos^2 x}\left(\dfrac{1}{3x^2}+\dfrac{1}{4}+\dfrac{2}{\sin x}\right)$

(e) $\dfrac{(x^3+1)\sqrt{2x+3}}{\cos^2 x}\left(\dfrac{1}{3x^2}+2\sqrt{2x+3}-\dfrac{1}{2\sin x\cos x}\right)$

$\ln\left(\frac{(x^3+1)\sqrt{2x+3}}{\cos^2 x}\right) = \ln\big((x^3+1)\sqrt{2x+3}\big)\big) - \ln(\cos^2 x) =$
$\ln\big(x^3+1\big) + \ln\big(\sqrt{2x+3}\,\big) - 2\ln\big(\cos(x)\big) = \ln\big(x^3+1\big) + \dfrac{1}{2}\ln(2x+3) - 2\ln\big(\cos(x)\big).$
$\dfrac{f'(x)}{f(x)} = \dfrac{3x^2}{x^3+1} + \dfrac{1}{2}\dfrac{2}{2x+3} - \dfrac{-2\cos(x)\sin(x)}{\cos^2(x)} = \dfrac{3x^2}{x^3+1} + \dfrac{1}{2x+3} + \dfrac{2\sin(x)}{\cos(x)}.$

3. Calculate the following indefinite integral.

$$\int \frac{x^2}{\sqrt{1-x^6}}dx\ .$$

(a) $\dfrac{\sin^{-1}(x^3)}{3}+C$ (b) $\sin^{-1}(x^3)+C$ (c) $\dfrac{\ln\left(\sqrt{1-x^6}\right)}{3}+C$

(d) $\dfrac{\tan^{-1}(x^3)}{3}+C$ (e) $\tan^{-1}(x^3)+C$

We know how to do $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$. If we do the substitute $u = x^3$ the denominator becomes $\sqrt{1-u^2}$. More over $du = 3x^2dx$ or $x^2dx = \dfrac{1}{3}du$ so $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}dx = \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1}(u) + C = \frac{1}{3}\sin^{-1}\big(x^3\big) + C.$

4. Compute $f'(x)$ if

$$f(x) = \tan^{-1}(\sqrt{1+e^{2x}})$$

(a) $\dfrac{e^{2x}}{(2+e^{2x})\sqrt{1+e^{2x}}}$ (b) $\dfrac{e^{2x}}{2(2+e^{2x})\sqrt{1+e^{2x}}}$ (c) $\dfrac{e^{2x}}{\sqrt{2+e^{2x}}\sqrt{1+e^{2x}}}$

(d) $\dfrac{2e^{2x}}{(2+e^{2x})}$ (e) $\dfrac{2e^{2x}}{(1+e^{2x})}$

By the Chain Rule

$$\frac{1}{1+\left(\sqrt{1+e^{2x}}\right)^2}\frac{d\sqrt{1+e^{2x}}}{dx} = \frac{1}{1+\left(\sqrt{1+e^{2x}}\right)^2}\frac{\frac{d1+e^{2x}}{dx}}{2\sqrt{1+e^{2x}}} = \frac{1}{1+\left(\sqrt{1+e^{2x}}\right)^2}\frac{2e^{2x}}{2\sqrt{1+e^{2x}}} =$$

$$\frac{1}{1+(1+e^{2x})}\frac{e^{2x}}{\sqrt{1+e^{2x}}} = \frac{e^{2x}}{(2+e^{2x})\sqrt{1+e^{2x}}}$$

5. Calculate the following indefinite integral.

$$\int \frac{e^{2x}+\sin x\cos x}{1+e^{2x}+\sin^2 x}dx \ .$$

(a) $\frac{1}{2}\ln|1+e^{2x}+\sin^2 x|+C$ (b) $2\ln|1+e^{2x}+\sin^2 x|+C$

(c) $\tan^{-1}(e^x+\sin x)+C$ (d) $\dfrac{1}{2(1+e^{2x}+\sin^2 x)^2}+C$

(e) $\dfrac{2}{(1+e^{2x}+\sin^2 x)^2}+C$

We need to get a better denominator so try $u = 1+e^{2x}+\sin^2 x$. Then $du = \big(2e^{2x} + 2\sin(x)\cos(x)\big)dx$ or $\frac{1}{2}du = \big(e^{2x}+\sin(x)\cos(x)\big)dx$ so

$$\int \frac{e^{2x}+\sin x\cos x}{1+e^{2x}+\sin^2 x}dx = \frac{1}{2}\int\frac{du}{u}$$

6. A radiologist injects a patient with 3/2 mCi of Technetium-99m(Tc) in preparation for a medical scan. The half life of Technetium-99m(Tc) is 6 hours. How much Technetium-99m(Tc) will remain in the patient's system after 9 hours have passed?

(a) $\dfrac{3}{4\sqrt{2}}$ mCi (b) $\dfrac{3\ln(2)}{12}$ mCi (c) $\dfrac{3}{4}$ mCi (d) $\dfrac{3}{4\ln 2}$ mCi (e) $\dfrac{3}{8}$ mCi

Let $A(t)$ be the amount of Technetium-99m in mCi at time t in hours. $A(t) = A_0 e^{-kt}$. From the half life result $A(6) = A_0/2$ and $A(6) = A_0 e^{-6k}$ so $e^{-6k} = \frac{1}{2}$. Hence $-6k = \ln\left(\frac{1}{2}\right) = -\ln(2)$ or $k = \frac{\ln(2)}{6}$.

$A(0) = A_0 = \frac{3}{2}$.

$$A(t) = \frac{3}{2}e^{-\frac{\ln(2)}{6}t} = \frac{3}{2}2^{-t/6} = 3 \cdot 2^{-1-t/6}$$

Hence

$$A(9) = 3 \cdot 2^{-1-9/6} = 3 \cdot 2^{-5/2} = \frac{3}{2^{5/2}} = \frac{3}{4\sqrt{2}}$$

7. Which of the following statements is true?
(Note: pay careful attention to the value of a in $\lim\limits_{x\to a}$.)

(a) $\lim\limits_{x\to\infty} 2^{-x} = 0$ (b) $\lim\limits_{x\to\infty} \tan^{-1}(x) = +\infty$ (c) $\lim\limits_{x\to\infty} \ln\left(\frac{1}{x}\right) = +\infty$

(d) $\lim\limits_{x\to 0} \sin^{-1}(x) = +\infty$ (e) $\lim\limits_{x\to 0} e^{1/x} = 1$

$\lim\limits_{x\to\infty} 2^{-x} = \lim\limits_{x\to\infty} \frac{1}{2^x} = \frac{1}{\lim\limits_{x\to\infty} 2^x} = \frac{1}{\infty} = 0.$

$\lim\limits_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}.$

$\lim\limits_{x\to\infty} \ln\left(\frac{1}{x}\right) = -\lim\limits_{x\to\infty} \ln(x) = -\infty.$

$\lim\limits_{x\to 0} \sin^{-1}(x) = \sin^{-1}(0) = 0.$

$\lim\limits_{x\to 0^+} e^{1/x} = e^{\infty} = \infty$ and $\lim\limits_{x\to 0^-} e^{1/x} = e^{-\infty} = 0$ so the limit does not exist.

8. Evaluate the limit

$$\lim_{x\to 1} \frac{\ln(x) + (x-1)^3}{1 - e^{(1-x)}}.$$

(a) 1 (b) 0 (c) -1 (d) ∞ (e) $-\infty$

The limit of the numerator is 0 and the limit of the denominator is 0 so we may apply l'Hospital's Rule. The derivative of the numerator is $\frac{1}{x} + 3(x-1)^2$. The derivative of the denominator is $e^{(1-x)}$.

$$\lim_{x\to 1} \frac{\ln(x) + (x-1)^3}{1 - e^{(1-x)}} = \lim_{x\to 1} \frac{\frac{1}{x} + 3(x-1)^2}{e^{(1-x)}} = \frac{1}{e^0} = 1$$

9. Evaluate the integral

$$\int_0^{\pi/6} x\cos(3x)\, dx.$$

(a) $\frac{\pi-2}{18}$ (b) $\frac{\pi}{18}$ (c) $\pi - 1$ (d) $\frac{\pi+2\sqrt{3}}{36}$

(e) $\frac{\pi+2\sqrt{3}-4}{36}$

First do $u = 3x$, $du = 3dx$; $\frac{1}{3}du = dx$.

$$\int_0^{\pi/6} x\cos(3x)\, dx = \frac{1}{3}\int_0^{\frac{\pi}{2}} u\cos(u)\frac{1}{3}du = \frac{1}{9}\int_0^{\frac{\pi}{2}} u\cos(u)du$$

Parts $dw = \cos(u)du$; $v = u$: $w = \sin(u)$; $dv = du$.

$$\int_0^{\frac{\pi}{2}} u\cos(u)du = u\sin(u)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin(u)du = \frac{\pi}{2} + \cos(u)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1$$

and

$$\int_0^{\pi/6} x\cos(3x)\, dx = \frac{1}{9}\left(\frac{\pi-2}{2}\right) = \frac{\pi-2}{18}$$

10. Find $\int_0^{\pi/2} \sin^4(x)\ \cos^3(x)\ dx.$

(a) $\dfrac{2}{35}$ (b) $\dfrac{4}{45}$ (c) $\dfrac{\pi^5}{5(2^5)} - \dfrac{\pi^7}{7(2^7)}$ (d) $-\dfrac{4}{45}$

(e) $\dfrac{1}{30}$

Use $u = \sin(x)$ since the power of the cosine is odd.

$$\int_0^{\pi/2} \sin^4(x)\ \cos^3(x)\ dx = \int_0^{\pi/2} \sin^4(x)\ \cos^2(x)\ \cos(x)\ dx =$$

$$\int_0^{\pi/2} \sin^4(x)\big(1-\sin^2(x)\big)\ \cos(x)\ dx = \int_0^1 u^4(1-u^2)du = \int_0^1 (u^4-u^6)du =$$

$$\left(\frac{u^5}{5}-\frac{u^7}{7}\right)\Bigg|_0^1 = \left(\frac{1}{5}-\frac{1}{7}\right) - 0 = \frac{7-5}{35} = \frac{2}{35}.$$

11. Compute the following integral

$$\int \sqrt{9-x^2}\,dx$$

(You may find the trigonometric formulas at the end of the exam helpful throughout.)

Write your answer in terms of the original variable x.
(Eliminate all composite trigonometric functions of the form $\sec\big(\tan^{-1}(x/n)\big)$, $\tan\big(\sec -1(x/n)\big)$, $\cos\big(\sin^{-1}(x/n)\big)$, $\tan(\tan^{-1}(x/n))$, $\sin(\sin^{-1}(x/n))$ etc..., n an integer, from your answer, and replace them by algebraic combinations of x (such as $ax+b$ or $\dfrac{\sqrt{ax^2+bx+c}}{x}$, where a, b, and c are constants). Non-composite trigonometric functions such as $\sin^{-1}(x/n)$ may be included in your answer.)

Before doing a trig. substitution we need to massage the integrand into a

$$\int \sqrt{9-x^2}\,dx = 3\int \sqrt{1-(x/3)^2}\,dx$$

Then $\sin(\theta) = \dfrac{x}{3}$ so $\cos(\theta)d\theta = \dfrac{dx}{3}$ or $3\cos(\theta)d\theta = dx$ and $\sqrt{1-(x/3)^2} = \sqrt{1-\sin^2(\theta)} = \cos(\theta)$. Hence

$$\int \sqrt{9-x^2}\,dx = 3\int \sqrt{1-(x/3)^2}\,dx = 3\cdot 3\int \cos^2(\theta)d\theta = 9\int \frac{1+\cos(2\theta)}{2}d\theta =$$

$$\frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

We need to rewrite in terms of functions of x.

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\dfrac{x}{3}\left(\sqrt{1-\left(\dfrac{x}{3}\right)^2}\right) = \dfrac{2}{9}x\sqrt{9-x^2}$. Then

$$\int \sqrt{9-x^2}\,dx = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C$$

Remark: Mathematica™ gives the solution as

$$\frac{x\sqrt{9-x^2}}{2} - 9\tan^{-1}\left(\frac{\sqrt{9-x^2}}{3+x}\right) + C$$

12.

(a) Find the partial fraction decomposition of

$$\frac{4x^2+x+2}{x(x^2+1)}.$$

(Solve for the constants (usually denoted by A, B, C etc...) in the decomposition.)

The degree of the numerator is 2 and the degree of the denominator is 3 so

$$\frac{4x^2+x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$4x^2+x+2 = A(x^2+1) + (Bx+C)x$$

Plug in $x=0$; $2=A$. Equating coefficients of x^2: $4 = A+B$ and $A=2$ so $B=2$. Equating coefficients of x: $1=C$, so

$$\frac{4x^2+x+2}{x(x^2+1)} = \frac{2}{x} + \frac{2x+1}{x^2+1}$$

(b) (b) Integrate $\displaystyle\int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{x+1}{x^2+1}\right)dx.$

Note: This is already decomposed into partial fractions and is unrelated to Part (a).

$$\int \left(\frac{1}{x}+\frac{2}{x^2}+\frac{x+1}{x^2+1}\right) dx = \int \frac{1}{x}dx + \int \frac{2}{x^2}dx + \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$
$$= \ln|x| - \frac{2}{x} + \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + C,$$

using substitution to compute $\int \frac{x}{x^2+1}dx$.

13. Find

$$\lim_{x\to 0}(1+2\sin x)^{1/x}$$

$(1+2\sin x)^{1/x} = e^{\ln((1+2\sin x)^{1/x})}$.

Compute $\lim_{x\to 0}\ln((1+2\sin x)^{1/x}) = \lim_{x\to 0}\frac{\ln(1+2\sin x)}{x}$, Using L'Hospital, we see that this $= \lim_{x\to 0}\frac{2\cos x/(1+2\sin x)}{1} = 2$. Therefore

$$\lim_{x\to 0}(1+2\sin x)^{1/x} = e^2$$

14. You will be awarded these two points if your name appears in CAPITALS on the Front of your exam and you mark your answers on the front page with an X through your answer choice like so: (a) (certainly **not** an O around your answer choice) . You may also use this page for

ROUGH WORK

The following is the list of useful trigonometric formulas:

Note: $\sin^{-1} x$ and $\arcsin(x)$ are different names for the same function and $\tan^{-1} x$ and $\arctan(x)$ are different names for the same function.

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x - y) + \sin(x + y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x - y) - \cos(x + y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x - y) + \cos(x + y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

There are currently 10 multiple choice problems.
There are currently 4 partial credit problems.